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On Extending the Domain of Definition of Čebyšev and Weak Čebyšev Systems

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1. INTRODUCTION AND STATEMENT OF RESULTS

In what follows, M will denote a set of real numbers having at least $n+1$ points, I will denote a real interval, $F(M)$ the set of real functions on M, $C(I)$ the set of continuous real functions on I, and S an n-dimensional subspace of $F(M)$. If J is a subset of M, the restriction of S to J is the set of all restrictions to J of elements of S. If $Z_n = \{z_1, ..., z_n\}$ is a subset of $F(I)$, we shall say that Z_n is a *Čebyšev system* (weak Čebyšev system) if for every set $\{t_1, ..., t_n\} \subseteq M$ such that $t_1 < t_2 < \cdots < t_n$, $\det\{z_i(t_j);\ i, j = 1, ..., n\} > 0$ ($\geqslant 0$). If Z_k is a (weak) Čebyšev system for $k = 1, ..., n$, we say that Z_n is a (weak) *Markov system*, or a complete (weak) Čebyšev system. The linear span of a (weak) Čebyšev system will be called a (weak) *Haar space*, and the linear span of a (weak) Markov system will be called a (weak) *Markov space*. If $z_1 \equiv 1$ then Z_n, as well as its linear span, will be called *normalized*. This terminology is consistent with that used by Karlin and Studden [1], and is somewhat more restrictive than the one used by Zielke [2]. The motivation for the term "normalized" is that if $\{z_1, ..., z_n\}$ is a Markov system, then $\{1, z_2/z_1, ..., z_n/z_1\}$ is a Markov system as well.

In this paper, we study the problem of extending the domain of definition of Haar or weak Haar spaces, but we must first introduce some additional definitions that will be used in the sequel.

If S is an n-dimensional (weak) Haar (or Markov) space defined on a set

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M, and $a := \inf(M) > -\infty$, we say that S can be *continued to the left*, provided that there is an n-dimensional (weak) Haar (or Markov) space U defined on a set of the form $(d, a] \cup M$, with $d < a$, such that the restriction of U to M coincides with S. Continuation to the right is defined similarly. If S is an n-dimensional subspace of $F(M)$ we say that S is a (weak) *E-space* if it has a basis $Z_n = \{z_1, ..., z_n\}$ such that for any integers $1 \leqslant r(1) < \cdots < r(m) \leqslant n$, $\{z_{r(k)};\ k = 1, ..., m\}$ is a (weak) Markov system on I. We shall also say that Z_n is a (weak) *E-system*. Thus an E-system is a type of Descartes system [1], a D_+-system in the terminology of Krein and Nudel'man [3] (see also Remark 2 at the end of Section 1). Finally, if $z_1 \equiv 1$ we say that S is a (weak) NE-space, and Z_n is a weak *NE-system*.

Following Zielke [2], we say that M has property (B) provided that between any two elements of M is a third element of M.

In this paper we prove the following:

THEOREM 1. *Let S be an n-dimensional Haar space defined on a set M having property* (B), *and such that* $\inf(M) > -\infty$. *Assume moreover that if an endpoint of M belongs to M, then it is a point of accumulation of M and all the functions in S are continuous there. Then the following propositions are equivalent*:

a. *S can be continued to the left to a Markov space.*

b. *S is an E-space.*

COROLLARY 1. *Let S be an n-dimensional Haar space of continuous functions defined on an open interval $I = (a, b)$ such that $a > -\infty$. Then*:

a. *S can be continued to the left if and only if S is an E-space.*

b. *S can be continued to the left to a space of continuous functions if and only if S contains an E-system $Z_n = \{z_1, ..., z_n\}$ such that* $\lim_{x \to a^+} z_1(x) > 0$.

The questions addressed in this paper, that of extending the domain of definition of a Haar space and that of finding bases which are Markov systems or E-systems, were apparently first considered by S. N. Bernstein. Bernstein introduced the notion of Descartes system in 1926 and claimed in 1938 to have shown that every Haar subspace of $C[a, b]$ has a basis that is an E-system on (a, b). This statement is false, as simple examples show; however, in 1972 V. S. Videnskii showed that a similar statement holds when additional conditions (including sufficient differentiability) are imposed (cf. [3]).

Krein and Nudel'man [3] attribute the first example of a noncontinuable Haar space to V. I. Volkov (1958), and show that if the domain of an n-dimensional Haar space may be extended by $n-1$ points then it has a basis which is a Markov system. This fact is behind most examples of noncontinuable Haar spaces (see, e.g., [2]). The problem of when a Markov space may be continued is left in [3] as an open question. In addition to those above, Zielke, Németh, and the authors have considered the problems at hand. In particular, if S is a Haar space, the problem of extending its domain of definition by a finite number of points has been studied by A. B. Németh [4, 5] and by Zielke (cf. [2]). For instance, it follows from [4, Theorem 2] that if Z_n is a set of continuous functions on $(a, b]$ that is a Čebyšev system on (a, b) and, moreover, not all the functions z_i vanish at b, then Z_n is a Čebyšev system on $(a, b]$ as well.

Corollary 1 was first conjectured by Zalik [6] in 1974. He gave a proof based on an integral representation of weak Markov systems that was later shown to be incorrect. It was also conjectured independently by D. Zwick in 1980. In several lectures since then he has proposed a solution based on generalized divided differences. Using this method, he proved the assertion for $n=3$, and indicated an inductive argument for the general case. Statements similar to Corollary 1 and the sufficiency part of Theorem 1 have been proved independently and simultaneously by Sommer and Strauss [7], using a different method.

Let $-M$ denote the set of all points t such that $-t$ is in M, and let S^- denote the space of all functions $f(t)$ such that $f(-t)$ is in S. We also have:

COROLLARY 2. *Let $-M$ and S^- satisfy the hypotheses of Theorem* 1. *Then S can be continued to the right if and only if S^- is an E-space on $-M$.*

A finite-dimensional space S contained in $F(M)$ is said to be *endpoint nondegenerate* (END) provided that for every c in M, the restrictions of S to $M \cap (-\infty, c)$ and to $M \cap (c, \infty)$ have the same dimension as S. This term, coined by D. J. Newman in 1980, was first used by Zwick in [8]. It was also used by Zielke in [9], where it is referred to simply as "nondegeneracy." We have:

THEOREM 2. *Let S be an END normalized weak Markov space defined on a set M such that* $\inf(M) > -\infty$. *Assume moreover that if an endpoint belongs to M then it is a point of accumulation of M and all the functions in S are continuous there. Then the following propositions are equivalent*:

a. *S can be continued to the left to an END normalized weak Markov space.*

b. *S is a weak NE-space.*

We also have:

COROLLARY 3. *Let $-M$ and S^- satisfy the hypotheses of Theorem* 2.

Then S can be continued to the right to an END normalized weak Markov space if and only if S^- is a weak NE-space in $-M$.

2. Proofs

Proof of Theorem 2. Let $a = \inf(M)$, and assume that S can be continued to the left to a weak NE-space S_0 defined on $M_0 := (d, a] \cup M$, for some $d < a$. According to [9, Theorem 3], there exists a basis $G_n = \{g_1, ..., g_n\}$ of S_0, with $g_1 \equiv 1$, a strictly increasing function h in $F(M_0)$, continuous functions $w_2, ..., w_n$, defined and increasing on $(\inf h(M_0), \sup h(M_0))$, and a point c in $h(M_0)$, such that for all x in M_0, and $k = 2, ..., n$,

$$g_k(x) = \int_c^{h(x)} \int_c^{t_2} \cdots \int_c^{t_{k-1}} dw_k(t_k) \cdots dw_2(t_2). \tag{1}$$

Although in his results Zielke asserts the existence of *some* c for which (1) is satisfied, an inspection of his proofs shows that a representation of the form (1) exists for *any* c in $h(M_0)$. This can also be verified directly: A representation such as (1) for another c' may be obtained directly from (1) by a triangular linear transformation; i.e., there is a basis G_n^0 of S_0, obtained from G_n by a transformation of the form $g_1^0 \equiv g_1$, $g_k^0 = g_k + \sum_{r=1}^{k-1} a_{r,k}\, g_r$, having a representation of the form (1) with c replaced by c'. In particular, we may assume that $h(d^+) < c < h(a)$. We now show that G_n is a weak E-system on M. Let integers $1 \leqslant r(1) < \cdots < r(N) \leqslant n$ be given, and assume, for instance, that $r(1) > 1$. Then

$$g_{r(k)}(x) = \int_c^{h(x)} q_{r(k)}(t)\, dw_2(t), \qquad \text{where} \quad q_2(x) \equiv 1,\ q_3(x) = \int_c^x dw_3(t),$$

and

$$q_k(x) = \int_c^x \int_c^{t_2} \cdots \int_c^{t_{k-2}} dw_k(t_{k-1}) \cdots dw_3(t_2), \qquad k = 4, 5, ...$$

Since termwise integration of a weak Markov system yields a weak Markov system, as readily follows from [3, p. 40], and for every x in M, $h(x)$ is larger than c, by an inductive procedure involving the number of integrations, we infer that $\{g_{r(1)}, ..., g_{r(N)}\}$ is a weak Markov system on M_0, whence the conclusion follows.

Conversely, assume that S is a weak NE-space on M and let $Z_n = \{z_1, ..., z_n\} \subseteq S$ be a weak NE-system.

We shall prove the assertion by induction on n. Since z_2 must be nonnegative and increasing, the validity of the assertion for $n=2$ is obvious. We shall now prove the inductive step.

If $V_n=\{v_1, ..., v_n\}$ is a set of real-valued functions defined on a real set M_1 we say that Z_n can be *embedded* in V_n if there is a strictly increasing function $h: M \to M_1$ such that $v_i[h(t)]=z_i(t)$ for every t in M $(i=1, ..., n)$. The function h is called an *embedding function.*

Since the functions z_k are increasing, it is clear that for any interval $[a, b]$, $\inf(M)<a<b<\sup(M)$, they are bounded in $[a, b] \cap M$. Let Z_n^0 denote the restriction of Z_n to $M_1 := M - \{\inf(M), \sup(M)\}$. Repeating verbatim the procedure employed in the proof of the theorem of [10] we conclude that Z_n^0 can be embedded in a normalized weak Markov system U_n of continuous functions defined on a bounded open interval (c, d) such that if h is the embedding function then $h(\inf(M)^+)=c$, $h(\sup(M)^-)=d$. An inspection of the proof also reveals that U_n must be an *END* weak *NE*-system. Thus, it suffices to show that U_n can be extrapolated to the left (that is, all of the functions in U_n can be extrapolated to the left). (Note: There is a typographical error in the definition of α_j and β_j in [10]. They should be defined as follows: $\alpha_j=2^{-j}$ if $|u_{r+1}(t_j^+)-u_{r+1}(t_j)|>0$, and 0 otherwise, and $\beta_j=2^{-j}$ if $|u_{r+1}(t_j)-u_{r+1}(t_j^-)|>0$, and 0 otherwise. This ensures the convergence of the series $\sum_{t_j<t}(\alpha_j+\beta_j)$.) Since u_2 is increasing, $(c, d)=A \cup B$, where $B=\bigcup [c_i, d_i)$, and u_2 is strictly increasing on A and constant on each interval $[c_i, d_i)$ (note that since U_n is END u_2 cannot be constant on an interval of the form $(c, c+\varepsilon)$ or $(d-\varepsilon, d)$ with $\varepsilon>0$). Moreover, [9, Lemma 3] implies that the functions u_k are constant on each interval $[c_i, d_i)$. We now remove B and "close the gaps" in A. Formally we proceed as follows: Let $q: A \to R$ be defined by $q(t)=t-\sum_{c_i<t}(d_i-c_i)$; then q is strictly increasing and $q(A)=(c, e)$ for some real number e. Thus, if $r_k(t) := u_k[q^{-1}(t)]$, then $R_n=\{r_1, ..., r_n\}$ is an END weak *NE*-system on (c, e), and $r_2(t)$ is strictly increasing and continuous there. Setting $v_k=r_k \circ r_2^{-1}$, we see that $V_n=\{v_1, ..., v_n\}$ is an END weak *NE*-system defined on a bounded open interval I with $v_1(x)=1$ and $v_2(x)=x$. It suffices to show that V_n can be extrapolated to the left.

Since the functions $v_k(x)$ are increasing on I, they are differentiable on a set D dense in I. Moreover, since $\{1, x, v_k(x)\}$ is a weak Markov system for $k=3, ..., n$, the functions v_k are convex [1, p. 376] and therefore continuous on I.

Let $1 \leqslant r(1)< \cdots <r(m) \leqslant n$ be given. If $r(1)=1$, since $\{v_{r(k)}; k=1, ..., m\}$ is a normalized weak Markov system, [11, Lemma 1] implies that $\{v'_{r(k)}; k=2, ..., m\}$ is a weak Markov system on D. Otherwise set $r(0)=1$, and proceeding in a similar fashion we conclude that $\{v'_{r(k)}; k=1, ..., m\}$ is weak Markov system. We have thus shown that $\{v'_2, ..., v'_n\}$ is a weak *NE*-system on D. Moreover, it is also END, whence by the inductive hypothesis the system $\{v'_2, ..., v'_n\}$ has a left extrapolation

on D. Since the functions v_k are convex they are absolutely continuous in any closed subinterval of I. Thus, if c is an arbitrary but fixed point in I, there is a sequence $\{c_k; k=1, ..., n\}$ such that

$$v_k(x) = c_k + \int_c^x v_k'(t)\, dt, \qquad k=1, ..., n,$$

and by another application of [3, p. 40] we deduce that the system V_n has a left extrapolation on D, from which the conclusion readily follows.

Q.E.D.

Proof of Theorem 1. Let $a = \inf(M)$. Assume that S can be continued to the left to a Haar space S_0 defined on $M_0 := (d, a) \cup M$ for some $d < a$. The results of [12] or [13] imply that S is a Markov space on $M_1 := M_0 - \sup(M_0)$. Using [9, Corollary 3] and proceeding as in the proof of Theorem 2, we deduce that there is a set $U_n = \{u_1, ..., u_n\} \subseteq S$ that is an E-system on M_1. Since the functions u_k, $k=2, ..., n$, must be positive and strictly increasing on M_1, the hypotheses imply that if $\sup(M_0) \in M_0$, then the functions u_k must be positive at this point. The conclusion now follows by an application of [2, Theorem 11.2].

Conversely, assume that S is an E-space, and let $Z_n = \{z_1, ..., z_n\}$ be an E-system in S on M. Set $v_k = z_k/z_1$, and $V_n = \{v_1, ..., v_n\}$. Since z_1 can clearly be extrapolated to the left as a strictly positive function, it suffices to show that V_n can be extrapolated to the left. Since V_n is an NE-system on M, and therefore a weak NE-system there, Theorem 2 ensures that V_n can be extrapolated to the left to a normalized weak Markov system U_n defined on $M_0 := (d, a) \cup M$. Applying [9, Theorem 3], we conclude that there exists a basis $G_n = \{g_1, ..., g_n\} \subseteq \operatorname{span}\{U_n\}$, with $g_1 \equiv 1$, a strictly increasing function h in $F(M_0)$, continuous functions $w_2, ..., w_n$, defined and increasing on $(\inf h(M_0), \sup h(M_0))$, and a point c in $h(M_0)$, such that the functions $g_k(x)$ have a representation of the form (1) on M_0. This implies that $h(t)$ must be bounded at a, and the functions $w_k(t)$ must be bounded at $\alpha := \inf h(M)$. It is also evident that the functions $w_k(t)$ must be strictly increasing on $h(M)$. Bearing in mind the remarks made following formula (1), there is no loss of generality in assuming $c \in h(M)$. Thus, setting

$$q(x) = h(x) \text{ on } M,\ q(x) = x - a + \lim_{t \to a^+} h(t), \text{ if } x \leqslant a,\ p_k(x) = w_k(x)$$

$$\text{on } (\alpha, \sup(h(M)), p_k(x) = x - \alpha + \lim_{t \to \alpha^+} w_k(x) \text{ if } x \leqslant \alpha,$$

and

$$u_k(x) = \int_c^{q(x)} \int_c^{t_2} \cdots \int_c^{t_{k-1}} dp_k(t_k) \cdots dp_2(t_2),$$

the conclusion readily follows. Q.E.D.

Remark 1. Note that in the proof of Theorem 2 we actually show that a normalized weak Markov system can be extrapolated to the left as a normalized weak Markov system iff it can be obtained from a weak *NE*-system by a triangular linear transformation. A similar statement applies to Theorem 1.

Remark 2. If $Z_n := \{z_1, ..., z_n\}$ is a Descartes system on M (that is, every nonempty subset of Z_n spans a Haar space), then Z_n need not be an E-system. However, it is true that $\operatorname{span}(Z_n)$ has a basis which is an E-system. To prove this, we adopt the following notation: For any square matrix A,

$$A\begin{pmatrix} i_1, & i_2, & ..., & i_p \\ j_1, & j_2, & ..., & j_p \end{pmatrix}$$

will stand for the determinant obtained from A by deleting all rows and columns except those labeled $i_1, i_2, ..., i_p$ and $j_1, j_2, ..., j_p$, respectively. Let $t_1 < \cdots < t_n$ be a fixed set of points in M and define a (nonsingular) matrix A by $a_{ij} = z_i(t_j)$ $(i, j = 1, ..., n)$. Set

$$v_i := \sum_{j=1}^{n} a_{ji} z_j \qquad (i = 1, ..., n),$$

assume that $x_1 < \cdots < x_n$ is an arbitrary set of points of M, and let $V := (v_i(x_j);\ i, j = 1, ..., n))$, $Z := (z_i(x_j);\ i, j = 1, ..., n))$. Note that the minors of Z of order k have fixed sign depending only on k, and the same sign as the corresponding minors of A. Thus, by the Cauchy–Binet formula ([14, p. 1])

$$V\begin{pmatrix} i_1, ..., i_k \\ j_1, ..., j_k \end{pmatrix} = \sum_{1 \leqslant l_1 < \cdots < l_k \leqslant n} \cdots \sum A\begin{pmatrix} l_1, ..., l_k \\ i_1, ..., i_k \end{pmatrix} Z\begin{pmatrix} l_1, ..., l_k \\ i_1, ..., i_k \end{pmatrix} > 0$$

for all $k = 1, ..., n$ and $1 \leqslant i_1 < \cdots < i_k \leqslant n$. Thus, $\{v_1, ..., v_n\}$ is an E-system. A similar statement holds for weak Descartes systems.

3. Examples

As mentioned in Section 1, examples of Haar spaces that may not be continued because they have no Markov basis abound (cf. [2, 3]). In this section we provide examples illustrating the concepts discussed above. To our knowledge, similar examples do not exist in the literature.

Example 1. A continuous normalized Markov system on a closed interval that cannot be extrapolated to a larger interval as a Markov system: Define $z_1(x) :\equiv 1$, $z_2(x) := x$, and $z_3(x) := -(x)^{1/2}$ for $x \in [0, 1]$. Then $Z_3 := \{z_1, z_2, z_3\}$ is a Markov system, since z_3 is strictly convex. Suppose that, for some $\varepsilon > 0$, Z_3 may be extrapolated to $[-\varepsilon, 1]$ so as to remain a Markov system. Then for the extrapolated functions, $z_1 > 0$, z_2/z_1 is strictly increasing and $(z_3/z_1) \circ (z_2/z_1)^{-1}$ is strictly convex on $(z_2/z_1)([-\varepsilon, 1]) =: [-\delta, 1]$. However, no function as $(z_3/z_1) \circ (z_2/z_1)^{-1}$, agreeing with z_3 on $[0, 1]$, can be convex on $[-\delta, 1]$, since $z_3'(0^+) = -\infty$. We note, however, that Z_3 may be transformed by a change of basis into a Markov system that can be extrapolated, e.g., $\{1, x^{1/2}, x\}$ is such a system. It is also possible to show that Z_3 cannot be extrapolated to the left as an END weak Markov system.

EXAMPLE 2. A Markov space containing constants with no basis that is a normalized Markov system: Let $z_1(x) := 2 - x^2$, $z_2(x) := xz_1(x)$, $z_3(x) :\equiv 1$, for $x \in [-1, 1]$. Then $Z_3 = \{z_1, z_2, z_3\}$ is a Markov system, since $z_1(x) > 0$, $(z_2/z_1)(x) = x$, and $(z_3/z_1)(x) = (2 - x^2)^{-1}$ is strictly convex on $[-1, 1]$. The linear span of Z_3 is, therefore, a normalized Markov space. However, one may easily check that no element of this space is strictly increasing; hence the linear span of Z_3 cannot contain a Markov basis, because if $\{1, v_2, v_3\}$ is such a basis, then v_2 must be strictly increasing.

EXAMPLE 3. A Markov space on an open interval that contains no basis that is an E-system: Let $z_1(x) := 1 - |x|$, and $z_2(x) := x$, for $x \in (-1, 1)$. Note that $z_1(x) > 0$ and that no nontrivial linear combination of z_1 and z_2 has more than one zero in $(-1, 1)$. This implies (see [2]), that $\{z_1, z_2\}$ is a Čebyšev system and, thus, a Markov system (one can also check the determinants). However, span$\{z_1, z_2\}$ does not contain an E-system on $(-1, 1)$ since every linear combination $a_1 z_1 + a_2 z_2$ with $a_2 \neq 0$ has a sign change in $(-1, 1)$, i.e., is not positive in $(-1, 1)$.

EXAMPLE 4. A weak Markov space on a closed interval with no basis that is a weak E-system: Let $z_1(x) := 0$ if $-2 \leqslant x < 0$, $z_1(x) := [1 - (x-1)^2]^{1/2}$ if $0 \leqslant x \leqslant 2$, and $z_2(x) := x$ for $x \in [-2, 2]$. Checking the appropriate determinants, we see that $\{z_1, z_2\}$ is a weak Markov system, on $[-2, 2]$. But, as in Example 3, any linear combination $a_1 z_1 + a_2 z_2$, with $a_1 \neq 0$, has a sign change in $(-2, 2)$, and thus cannot be nonnegative.

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